Self-organized criticality in dynamics without branching

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We demonstrate the phenomenon of self-organized criticality (SOC) in a simple random walk model described by a random walk of a myopic ant, i.e., a walker who can see only nearest neighbors. The ant acts on the underlying lattice aiming at uniform digging, i.e., reduction of the height profile of the surface but is unaffected by the underlying lattice. In one, two, and three dimensions we have explored this model and have obtained power laws in the time intervals between consecutive events of ''digging.'' Being a simple random walk, the power laws in space translate to power laws in time. We also study the finite size scaling of asymptotic scale invariant process as well as dynamic scaling in this system. This model differs qualitatively from the cascade models of SOC. [S1063-651X(98)03105-5]

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The concept of self-organized criticality (SOC) was introduced by Bak, Tang, and Wiesenfeld in the context of avalanches in a sandpile model (BTW model) [1]. A diffusively coupled spatially extended system which is driven adiabatically, i.e., the drive occurs only when the system has been fully relaxed, settles in the metastable state with very long correlations and no characteristic length scale. This model is termed to be self-organized since the critical state is reached, though no particular parameter seems to have been adjusted. There have been further variants of the above model which have similar rules, but are in different universality class [2]. The above models are cellular automata models in which the discrete variable value assigned to different points on a d-dimensional lattice are updated in discrete time [3]. The relevant perturbations in which SOC gets destroyed has been a topic of interest to many researchers [4]. Developing a partial differential equation model for SOC has also been an active area of interest [5]. There have been models with threshold dynamics in continuous variable values such as the adaptive dynamics model on coupled map lattices or earthquake models, though it is debatable whether the power laws arising in these models can be termed as self-organized [6,7].

In all these models SOC is induced by a branching process. The disturbance propagates from one length scale to the other by branching in various directions and this hierarchical basis for the dynamics leads to a power-law behavior. This description of branching leading to power laws has been given for diverse processes such as the intermittent turbulent process by Kolmogorov [8] or income distributions in the U.S. by Schlesinger [9]. However, scale invariant processes need not be produced by branching alone. The disturbance can choose a random direction yielding scale invariant structure. Here we propose a simple random walk model for SOC. As a physical illustration, we would like to note a recent experimental observation by Vishwanathan *et al.* [10] about the foraging behavior of sea birds. In this experiment, the authors studied the foraging behavior of wandering albatross. Measurements of the distance traveled by the bird at various times are carried out. They found a power-law behavior in distribution of flight time events. Interestingly, the observation is that though the distribution deviates significantly from simple random walk, it is still a power law implying a scale invariant manner in which the flights proceed. Assuming that the flight directions change randomly after finding food, they argued that the data they have suggests that the distribution of food on the ocean surface is also scale invariant. Although we do not attempt to model this experiment, it nicely illustrates the fact that not only the branching processes but the processes induced by a simple random walk or flight also can organize themselves in a scale invariant fashion in time and space. We would also like to note that continuous time random walks have been used to explain SOC in rice piles recently [11].

We introduce a model of self-organized scale invariant behavior in space and time which is induced by random walk. A model of Eulerian walkers (EW) has been introduced recently [12]. Our model is simpler in the sense that unlike the above model, the walker is unaffected by the medium. As will be clear in the course of discussion, not only it is in a different universality class, but is even qualitatively different from the earlier models.

Let us first discuss our model in one dimension for simplicity. We consider a lattice of length L. At each site i, $1 \leqslant i \leqslant L$, we associate an integer x_i which denotes the height of that point and $-\infty < x_i \leqslant 0$. To begin with, we assign $x_i = 0$ for all i. We put a random walker at a randomly chosen site j $(1 \leqslant j \leqslant L)$. Now the dynamics of the lattice is defined in the following way. (a) At each time step, the random walker moves to its nearest neighbor which is chosen randomly. (b) Before moving to the next site, the random walker compares the height at that site with those of nearest neighbors and reduces the height at that site by 1 unless any of the nearest neighbors has a higher height. In other words, if random walker is at site k, then

$$x_k = x_k - 1$$

unless $x_{k+1} > x_k$ or $x_{k-1} > x_k$. We will note this event of

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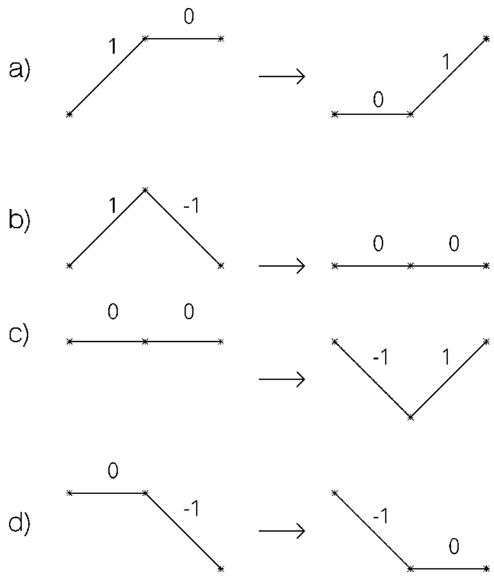

FIG. 1. Schematic diagram of the configurations that change by the action of the random walker which is at the center. Slopes are also shown.

reduction of height as ‘‘digging.’’ The condition above on digging is introduced since the aim of the random walker is to dig uniformly and it does not want to dig the site which already has a height lower than any of its neighbors. Though the medium is affected by the walk, the walker is unaffected by the medium, i.e., the next site to which the random walker moves is chosen randomly and is independent of the entire height profile. At boundaries the comparison is only one sided. If the random walker moves out of the lattice, we put it back in a randomly chosen site within the lattice. Since the random walker can not see beyond nearest neighbors we call the model the digging myopic ant (DMA) model. We can also describe the model in terms of the evolution rule for the slopes on either side of the random walker. (By construction the slopes can take only three values 1, 0, and -1.) Of the nine possible combinations, four of them transform as 1,0 $\rightarrow$0,1; 1,$-1\rightarrow$0,0; 0,0$\rightarrow -1$,1; 0,$-1\rightarrow -1$,0 while the other five remain unchanged. (See Fig. 1.) The rule at the left boundary is $0\rightarrow 1$; $-1\rightarrow 0$ and $1\rightarrow 1$ while at the right boundary $1\rightarrow 0$, $0\rightarrow -1$ and $-1\rightarrow -1$. Note that except at boundaries the sum of slopes remains conserved. We also note that the changes in boundary conditions do not change the results qualitatively. Digging by two units when the walker is on the hill (1,$-1\rightarrow -1$,1) also does not change results qualitatively, at least in 1D. We note that with this change the rules seem to be very similar to a traffic jam model with symmetry breaking [13]. However, since the evolution is highly spatially correlated in our case the properties are very different.

We start with a flat surface. This means that in the beginning all sites are potentially ‘‘active,’’ i.e., they can be dug. However, as the surface evolves, often a big valley or a big slanted surface appears. If one ignores the fact that the sites dug subsequently are not independent of each other, rather are spatially nearby, i.e., the noise in our case is correlated, one can relate the distribution of times required to reach active sites to the spatial distribution of active sites. Now we look at the distribution of time intervals between which active sites were visited.

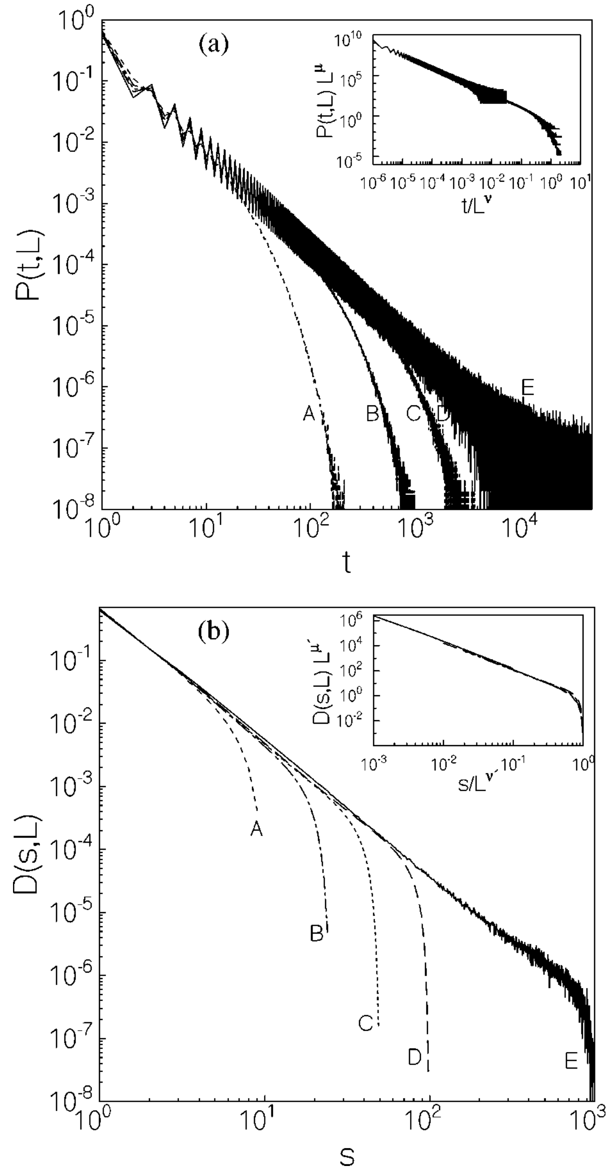

FIG. 2. (a) The interevent time distribution $P(t,L)$ vs time t in 1D. (b) Probability distribution $D(s,L)$ that s distinct sites are visited between two events vs s in 1D. [In both figures (A) $L=10$, (B) $L=25$, (C) $L=50$, (D) $L=100$, and (E) $L=1000$. Insets show finite-size scaling.] Here and in all the following figures the quantities plotted are dimensionless.

Here, the system is driven by random perturbations and the time interval t between two successive events of digging when the medium is affected by the walker is a quantity of interest. We compute the distribution $P(t)$ where $P(t)$ is the normalized probability that the time between two successive events of digging is t. We also compute the probability distribution $D(s)$, the number of distinct sites s visited by the random walker between two successive events. We find that $P(t)\sim t^{-\gamma}$, $\gamma\approx 1.6$. It is clear that the distribution $D(s)$ cannot be independent of $P(t)$ since in a simple random walk number of distinct sites s visited in time t goes as $t^{1/2}$. This implies $D(s)\sim s^{-\gamma'}$, $\gamma'=2\gamma-1$. (This is also verified in finite-size scaling defined later.) Thus as one would expect, a power-law distribution in time translates in a power-law distribution in space. Figures 2(a) and 2(b) show $P(t)$ and $D(s)$

for various lattice sizes in one dimension (1D). If one looks at the spatial profile of the lattice developed after a long time, one can see a surface which is far from uniform. Thus a myopic random walker who started the walk aiming at a uniform digging of the surface, ends up digging the surface in a scale invariant manner. Thus, unlike the BTW model, this model shows nontrivial nontransient scaling properties even in one dimension. However, we note that it has properties common with earlier SOC models. It is a conservative model except at the boundaries in the sense that the sum of slopes at all the sites does not change unless digging occurs at the boundary. As in earlier model, the boundary conditions are open. However, as seen above, evolution rule described in terms of local slopes is anisotropic. The relation with the distribution of active sites is not clear since noise is correlated.

Given the nature of the distributions, i.e., a simple power law followed by an exponential tail, one can fit a finite-size scaling form $P(t,L)=L^{-\mu}G(t/L^{\nu}),D(s,L)=L^{-\mu'}F(s/L^{\nu'})$ ($\mu=\gamma\nu$ and $\mu'=\gamma'\nu'$), to the distributions [14]. In 1D we can fit the scaling nicely with $\nu=2,\nu'=1$. This is useful in higher dimensions in particular where it is difficult to do a very large size simulations and scaling form gives the power-law exponents with reasonable accuracy. In Fig. 2 depicting the distributions $P(t)$ and $D(s)$, we also show the finite-size scaling in the inset.

The model can be easily extended to higher dimensions. We have studied this model in two and three dimensions. We plot interevent time distribution $P(t)$ in 2D and 3D in Figs. 3(a) and 4. As in 1D, $P(t)\sim t^{-\gamma}$ with $\gamma\approx1.2,\nu\approx2$ in 2D and $\gamma\approx1.2,\nu\approx1.8$ in 3D. Since the number of distinct sites covered s goes as $t/ln(t)$ in 2D and as t in 3D [15], one can expect a power-law distribution for $D(s)$ as well with $\gamma'=\gamma$. We numerically verified that $\gamma=\gamma'$. Of course, in 2D, we expect a logarithmic correction in the power-law form. Numerically, we observe that $D(s,L)=L^{-\mu'}F(s\ln(L)/L^{\nu'})$ with $\nu=\nu'$ gives a good fit. Figure 3(b) shows the distribution $D(s)$ in 2D. For 3D, site distribution was beyond our available computational resources. However, we expect it to closely follow the $P(t)$. In Figs. 3 and 4, the insets show the finite-size scaling in each of the cases as in Fig. 2. The geometrical picture in 2D is identical to that in 1D. One sees valleys of all sizes present in the asymptotic height profile in 2D. This is understandable. As in the sandpile model if one has a configuration with a single big valley, the random walker can go to the boundary and dig making sites in the interior active and thus one expects many events. (In our model, one more configuration in which not many sites will be active will be a long tilted interface. However, by the same logic, it will not stay for long.) Similarly, starting with a flat interface, one expects many events since all sites are active. Thus the surviving configuration, or the configuration which will be attained most of times will be the one in which valleys of all sizes are present.

We have also seen how the profile changes in time. The simplest quantitative measure that demonstrates the geometrical changes in the profile is roughness. The roughness $\sigma(L,t)$ of the interface of length L at time t (starting with a flat interface) is given by $\sigma(t,L)=\sqrt{(1/L)\Sigma_{i=1}^{L}(x_i(t)-\overline{x(t)})^2}$, where $\overline{x(t)}$ is the average

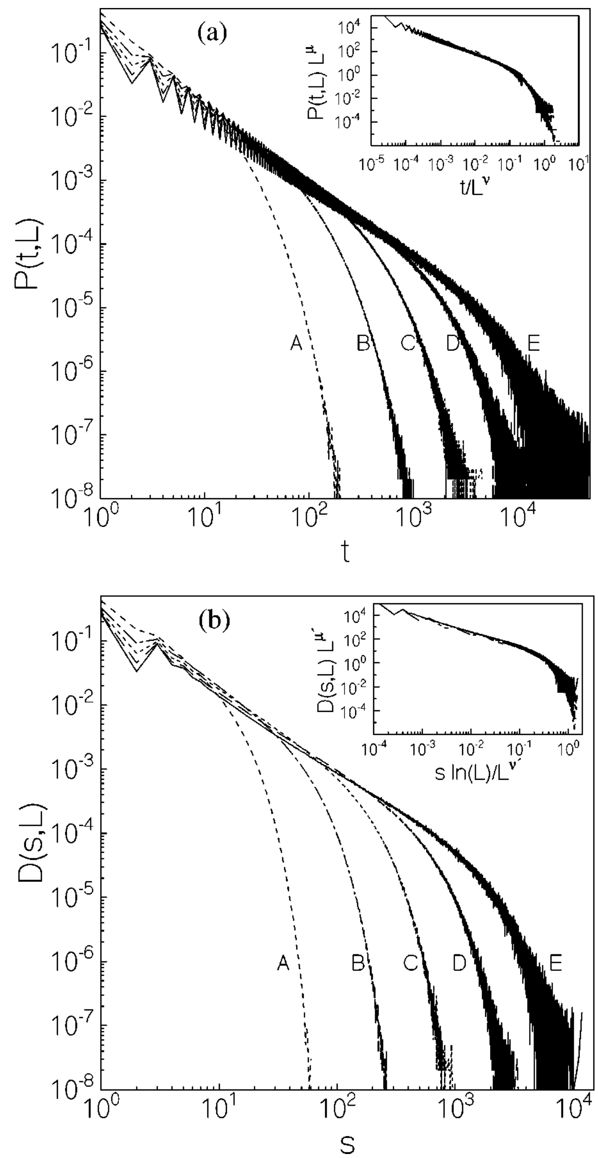

FIG. 3. (a) $P(t,L)$ vs time t in 2D. (b) $D(s,L)$ vs s in 2D. [In both figures (A) $L=10$, (B) $L=25$, (C) $L=50$, (D) $L=100$, and (E) $L=200$. Insets show finite-size scaling.]

height of the interface at time t. Growth depends on nearest neighbors and thus the correlations develop in time and span the entire length L. When the entire surface gets correlated the width saturates. The roughness $\sigma(L,t)$ follows a scaling relation $\sigma(t,L)=L^{\alpha}f(t/L^{z})$ (see, e.g., [16]). The exponent $z=\alpha/\beta+1$. Here we note that normally in growth models with sequential updates, one scales time t in units of L assuming that the entire interface gets updated after L time steps. We have not done so since it conflicts with our earlier notion of time. However, one can see that the above scaling with redefined time t will be same as well known Family-Vicsek scaling relation [16]. The exponent $\beta=0.565$ signifies the growth in time in the beginning ($\sigma(t,L)\sim t^{\beta}$), z gives saturation time ($t_{\rm sat}\sim L^{z}$) and $\alpha=1.1$ signifies saturation width ($\sigma_{\rm sat}\sim L^{\alpha}$). The scaling form with the above fit which assumes a power-law growth followed by saturation is reasonably good (see Fig. 5). For small times ($t<9, L\gg t$) one can easily compute all the possible configurations and their probabilities analytically. The values computed so are

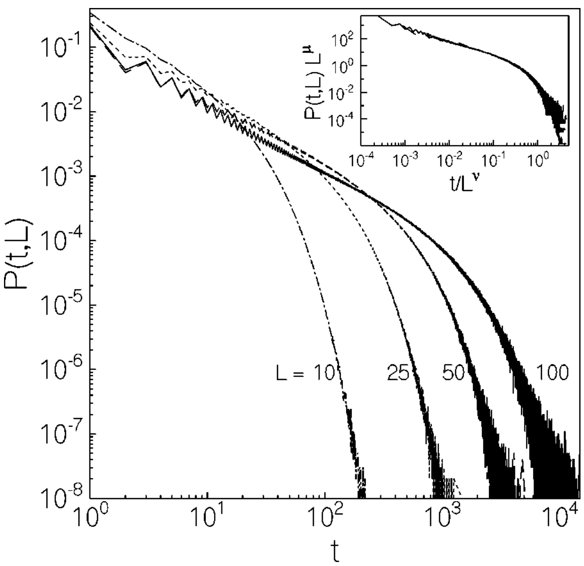

FIG. 4. $P(t,L)$ vs t in 3D. Inset shows finite-size scaling.

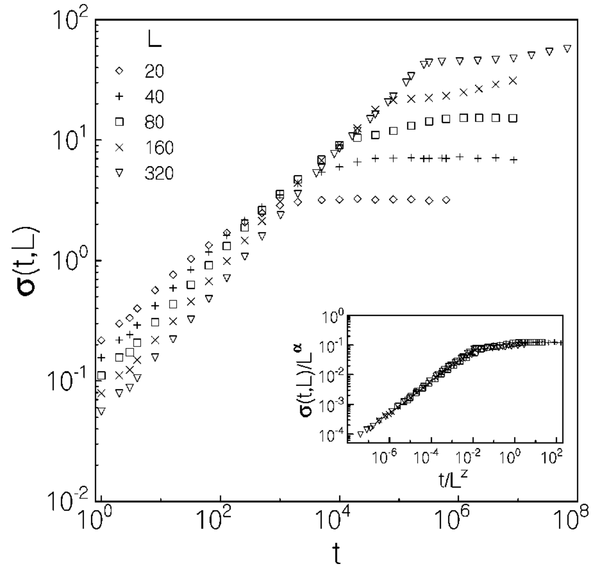

FIG. 5. Roughness $\sigma(t,L)$ vs time t for various L in 1D. Inset shows the dynamic scaling of the interface.

in close agreement with simulations and also yield the growth exponent $\beta = 0.565$. Large value of α reflects the highly inhomogeneous asymptotic interface.

We have also studied a variant of the model in which one tries to reduce the correlation between successive events by putting the random walker in a random position after each digging. Thus the noise is not spatially correlated any longer. Most of the qualitative features of the model do not change. The dynamic scaling in this variant and further investigations in the current model as well as its variant are deferred to a future publication.

In short, we have proposed a model of self-organized criticality in which the governing mechanism is that of diffusion. This model is hopefully easier to handle analytically since the exponents in space are easily related to exponents in time and one does not have a lot of unrelated and ill-understood exponents. We also feel that such models could be of use in situations which yield scale invariant behavior but do not involve cascades, but rather have diffusion as the only way in which information spreads in the system.

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